Frequency-Dependent Relaxation Time of Electron-Phonon Systems. II. Effects of Finite Lifetime of the Phonons

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(Received 21 August 1963)

In a previous paper a theory of the frequency-dependent relaxation time for a simple model of metal was developed, assuming the phonons to have infinite lifetime. In the present paper the problem is investigated from a more sophisticated point of view. The effects of low temperature are reconsidered and the effect of the finite lifetime of the phonons is included. The Green's function technique is used to obtain the response of the electron-phonon system to an oscillating electric field. Our treatment, which employs the time-dependent currents correlation function, does not have the time scale restriction of the usual transition probability approach, and does give a proper description of the time-dependent collective effects.

I. INTRODUCTION

IN a recent paper by one of us[1] a calculation of the frequency-dependent relaxation time was presented employing a kinetic-type approach. The treatment was developed in the following manner: The bare phonon was treated as a test particle whose energy spectrum is altered due to the interaction with the electrons. (Lifetimes of the dressed phonons were neglected.) Thereafter the conductivity was obtained using the interaction between electrons and the dressed phonons.

In the present paper the problem is reconsidered from a more sophisticated point of view. Here we calculate the frequency-dependent conductivity, from which one may compute the relaxation time, for a simplified model of a metal, composed of electrons and phonons, using the Green's function technique.

Using this technique, we are able to show that the change of the phonon spectrum is automatically included, provided we express our terms for the conductivity by a dressed phonon propagator. Moreover, we do not neglect electron correlations from the outset, a procedure adopted in I. However, it turns out, this correlation does not contribute to the high-frequency conductivity.

At high temperatures our results, apart from the effect of the finite lifetime of the phonons, are reduced to results of I. For low temperatures, i.e., temperatures much below the Debye temperature, the results of I may be obtained *only under the restriction* that the phonon frequency is taken to be much smaller than the field frequency, independent of the temperature. However, if a different limiting procedure is adopted, that is the temperature is taken to be the smallest parameter, our expression for the relaxation time is different.

II. THE GENERAL FORMALISM

We start with the general expression of the conductivity for a system of charged particles as given by Kubo[2] for wave number equal to zero. ($\hbar$ is taken to be one.)

$$\sigma(\omega) = \frac{1}{3V} \int_0^\infty d\tau e^{i\omega\tau} \int_0^\beta d\lambda \langle \mathbf{j}(\tau - i\lambda) \cdot \mathbf{j}(0) \rangle, \quad (1)$$

where ω is the frequency of the electromagnetic wave,

$$\mathbf{j}(\tau) = e^{iH\tau} \mathbf{j}(0) e^{-iH\tau} \quad (2)$$

is the Fourier transform of the electron current operator (the ion contribution to the current is neglected, due to their large mass) in the Heisenberg representation for wave number zero, and the average of an operator O is given by

$$\langle O \rangle = \mathrm{Tr}\{e^{\beta(\Omega + \mu N - H)} O\}. \quad (3)$$

In Eqs. (2) and (3), H represents the total Hamiltonian of the system, Ω is defined by

$$e^{-\beta\Omega} = \mathrm{Tr}\{e^{\beta(\mu N - H)}\}, \quad (4)$$

μ and N are the chemical potential and the number operator for the electrons, respectively, and β the inverse of the temperature in energy units. The current operator used in Eq. (2) is defined by

$$j(0) = \frac{e}{m} \sum_p \mathbf{p} a_p^\dagger a_p, \quad (5)$$

where we use the following convention for Fourier transforms:

$$f(\mathbf{x}, \tau) = \frac{1}{2\pi} \int d\omega \frac{1}{V} \sum_k \exp(-i\omega\tau - i\mathbf{k} \cdot \mathbf{x}) f(\mathbf{k}, \omega)$$

and

$$f(\mathbf{k}, \omega) = \int d\tau \int d\mathbf{x} \exp(i\omega\tau + i\mathbf{k} \cdot \mathbf{x}) f(\mathbf{x}, \tau). \quad (6)$$

[1] A. Ron, Phys. Rev. 131, 2041 (1963). We shall refer to this paper as I.

[2] J. M. Luttinger and J. C. Ward, Phys. Rev. 118, 1417 (1960). Many references on the Green's function technique may be found in this paper.

In order to render Eq. (1) in a more convenient form, we integrate it by parts and obtain

$$\sigma(\omega) = \sigma_0(\omega) + \sigma_1(\omega), \qquad (7)$$

where

$$\sigma_0(\omega) = i(\omega_p^2/4\pi\omega). \qquad (8)$$

Here $\omega_p = (4\pi e^2 n/m)^{1/2}$ is the plasma frequency; n, e, and m are, respectively, the electron density, charge, and mass; and

$$\sigma_1(\omega) = \frac{1}{3V\omega} \int_0^\infty d\tau e^{i\omega\tau} \langle [\mathbf{j}(\tau)\cdot,\mathbf{j}(0)] \rangle, \qquad (9)$$

where $[,]$ denotes the commutator.

We next define a Green's function

$$M(u) = (1/3V)\langle T\{\mathbf{j}(u)\cdot\mathbf{j}(0)\} \rangle, \quad -\beta < u < \beta, \qquad (10)$$

where T is the Dyson ordering operator and

$$\mathbf{j}(u) = e^{uH}\mathbf{j}(0)e^{-uH}. \qquad (11)$$

The function $M(u)$, defined in Eq. (10), is periodic in u, i.e.,

$$M(u+\beta) = M(u),$$

and thus its Fourier transform with respect to u is

$$M(\omega_n) = \int_0^\beta du e^{\omega_n u} M(u), \qquad (12)$$

where

$$\omega_n = 2\pi i n/\beta; \quad n = 0, \pm 1, \pm 2, \cdots. \qquad (13)$$

We now define $M(z)$ as the analytical continuation of $M(\omega_n)$ from the infinite set of points $2\pi i n/\beta$ $(n>0)$ on the positive imaginary axis of z to the entire upper half-plane of z. It is then easy to show[3] that

$$\sigma_1(\omega) = (1/i\omega)M(\omega + i\epsilon), \quad \epsilon \to 0_+.$$

We therefore obtain

$$\sigma(\omega) = \sigma_0(\omega) + (1/i\omega)M(\omega + i\epsilon), \quad \epsilon \to 0_+ \qquad (14)$$

as our expression for the absorption coefficient.

III. EVALUATION OF THE CONDUCTIVITY

We turn now to the calculation of $M(\omega_n)$ using a perturbation-expansion technique, and then resume all diagrams (terms) which contribute to the conductivity under the condition that the number of electrons in the Bohr sphere is large, the frequency is high compared to the collision frequency, the wavelength of the field is larger than the Bohr radius, and the electron-ion mass ratio is small. Thus, in resumming the diagrams (terms) of the perturbation expansion, we consider processes proportional to the number of the electrons, N, as finite, and include them to all orders; while those processes which are not proportional to N are treated as small.

We employ the Hamiltonian of Bardeen and Pines[3]

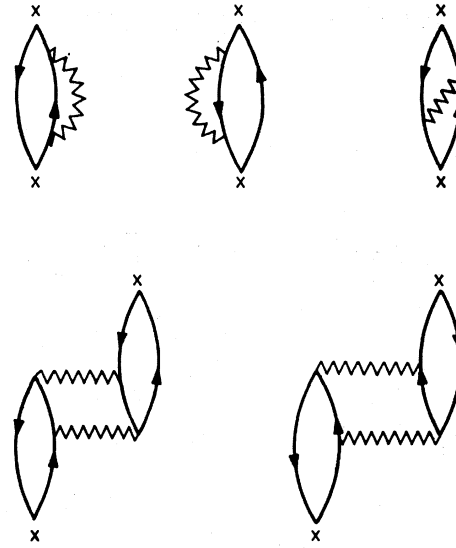[3] J. Bardeen and D. Pines, Phys. Rev. 99, 1140 (1955).

FIG. 1. The class of diagrams which contribute to the high-frequency conductivity.

as was redescribed in I by Eqs. (3)–(10), and also the notation of the latter paper. Our Green's function, Eq. (10), in the interaction representation is now given by[4]

$$M(u) = (1/3V)\langle U(\beta)\rangle_0^{-1}\langle T\{\mathbf{j}(u)\cdot\mathbf{j}(0)U(\beta)\}\rangle_0, \qquad (15)$$

where now $\mathbf{j}(u) = e^{uH_0}\mathbf{j}(0)e^{-uH}$ with H_0 the free-particle Hamiltonian, the symbol $\langle \ \rangle_0$ corresponds to an average with respect to H_0, and

$$U(\beta) = \exp\left\{ -\int_0^\beta du H_1(u) \right\}, \qquad (16)$$

with $H_1(u)$ the interaction Hamiltonian in the same representation.

The basic rules for the perturbation expansion of $M(\omega_n)$ and their diagrammatical representation are given essentially by Luttinger and Ward,[2] with the addition that here we also have an electron-electron interaction via a phonon. The essential ingredients of the perturbation expansion are: the free-electron propagator, given by

$$G_p(\zeta_l - E(\mathbf{p}))^{-1},$$

$$\zeta_l = (2l+1)\pi i/\beta + \mu; \quad l = 0, \pm 1, \pm 2, \cdots \qquad (17)$$

which is indicated diagrammatically by a solid line; the free-phonon propagator which is the Fourier transform of (Q_k is the phonon coordinate)

$$D_k(u) = \langle T\{Q_k(u)Q_k(0)\}\rangle \qquad (18)$$

or

$$D_k(\alpha_m) = 1/(\alpha_m - \Omega_k^2),$$

where Ω_k is the bare phonon frequency (see I),

$$\alpha_m = 2\pi i m/\beta; \quad m = 0, \pm 1, \pm 2, 111 \qquad (19)$$

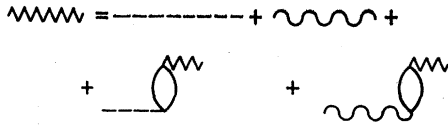[4] A. Ron and N. Tzoar, Phys. Rev. 131, 12 (1963).

FIG. 2. The integral equation for the effective interaction.

and is indicated by a wavy line; and the Coulomb interaction, indicated by a dashed line. To each electron-phonon vertex we assign, after Migdal,[5] v_k, i.e., each phonon line is associated with $|v_k|^2$.

In the present approximation we consider the class of diagrams displayed in Fig. 1, where the saw-toothed line represents the effective electron-electron potential shown in Fig. 2. The integral equation for the effective potential is

$$U_k(\alpha_m) = \varphi_k + |v_k|^2 D_k(\alpha_m) + \varphi_k Q_k(\alpha_m) U_k(\alpha_m)$$
$$+ |v_k|^2 D_k(\alpha_m) Q_k(\alpha_m) U_k(\alpha_m), \quad (20)$$

where φ_k represents the electron-electron Coulomb interaction (see I),

$$Q_k(\alpha_m) = \frac{1}{V}\sum_p \frac{1}{\beta}\sum_l G_{p+k/2}(\zeta_l + \alpha_m) G_{p-k/2}(\zeta_l)$$

$$= \frac{1}{V}\sum_p \frac{f_{p+k/2} - f_{p-k/2}}{E(p+k/2) - E(p-k/2) - \alpha_m} \quad (21)$$

and f_p is the Fermi distribution [see I, Eq. (20)]

$$f_p = \{\exp\beta[E(p) - \mu] + 1\}^{-1}, \quad (22)$$

and then the expression for the effective potential yields

$$U_k(\alpha_m) = \frac{\varphi_k + |v_k|^2 D_k(\alpha_m)}{1 - (\varphi_k + |v_k|^2 D_k(\alpha_m))Q_k(\alpha_m)}. \quad (23)$$

It is convenient to cast Eq. (23) into a form which shows explicitly the effect of dynamic screening of the electrons.

Define the electron dielectric function

$$\mathcal{E}_k(\alpha_m) = 1 - \varphi_k Q_k(\alpha_m) \quad (24)$$

and the "true" phonon propagator

$$\tilde{D}_k(\alpha_m) = \frac{D_k(\alpha_m)}{1 - [|v_k|^2 Q_k(\alpha_m)/\mathcal{E}_k(\alpha_m)]D_k(\alpha_m)}; \quad (25)$$

the "free" phonon propagator can be written as

$$D_k(\alpha_m) = \frac{\tilde{D}_k(\alpha_m)}{1 + [|v_k|^2 Q_k(\alpha_m)/\mathcal{E}_k(\alpha_m)]\tilde{D}_k(\alpha_m)} \quad (26)$$

and thus

$$U_k(\alpha_m) = \frac{\varphi_k}{\mathcal{E}_k(\alpha_m)} + \frac{|v_k|^2}{[\mathcal{E}_k(\alpha_m)]^2}\tilde{D}_k(\alpha_m). \quad (27)$$

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[5] A. B. Migdal, Zh. Eksperim. i Teor. Fiz. 34, 1438 (1958) [English transl.: Soviet Phys.—JETP 6, 996 (1958)].

The physical interpretation of Eq. (27) is rather simple. The first term represents the *dynamically screened* Coulomb potential while the second term represents a *dynamically* dressed phonon interacting with the electrons via a screened potential.

The contribution of the diagrams of Fig. 1 were essentially calculated in Refs. 3 and 6 and in our case yields, after tedious algebraic manipulations,

$$M(\omega_n) = \frac{e^2}{3m^2}\frac{1}{\omega_n^2}\frac{1}{(2\pi)^3}\int d\mathbf{k}\,k^2 \frac{|v_k|^2}{\varphi_k}\frac{1}{\beta}$$

$$\times \sum_m \tilde{D}_k(\alpha_m)\chi_k(\alpha_m + \omega_n)\left[\frac{1}{\mathcal{E}_k(\alpha_m + \omega_n)} - \frac{1}{\mathcal{E}_k(\alpha_m)}\right], \quad (28)$$

with

$$\omega_n = 2\pi n i/\beta; \quad n = 0, \pm 1, \pm 2, \cdots$$

and

$$\chi_k(\alpha_m) = 1 + \frac{|v_k|^2 Q_k(\alpha_m)}{\mathcal{E}_k(\alpha_m)}\tilde{D}_k(\alpha_m). \quad (29)$$

We now perform the summation over m and the analytical continuation of Eq. (28) by the same method employed in Refs. 4 and 7 and obtain

$$M(\omega) = \frac{e^2}{3m^2}\frac{1}{\omega^2}\frac{1}{(2\pi)^3}\int d\mathbf{k}\,k^2 \frac{|v_k|^2}{\varphi_k}\frac{P}{4\pi i}\int_{-\infty}^{+\infty} dx\,\coth\frac{\beta x}{2}$$

$$\times \left\{\left[\frac{1}{\mathcal{E}_k(x)} - \frac{1}{\mathcal{E}_k^*(x)}\right]\tilde{D}_k(x+\omega)\right.$$

$$+ [\tilde{D}_k(x) - \tilde{D}_k^*(x)]\frac{1}{\mathcal{E}_k(x+\omega)}$$

$$\left. - \left[\frac{\tilde{D}_k(x)}{\mathcal{E}_k(x)} - \frac{\tilde{D}_k^*(x)}{\mathcal{E}_k^*(x)}\right]\right\}, \quad (30)$$

where

$$\mathcal{E}_k(\omega) = 1 - \varphi_k\int \frac{d\mathbf{p}}{(2\pi)^3}$$

$$\times \frac{f_{p+k/2} - f_{p-k/2}}{E(p+k/2) - E(p-k/2) - \omega - i\delta}, \quad (31)$$

$$\tilde{D}_k(\omega) = \left[(\omega + i\delta)^2 - \Omega_k^2 - \frac{|v_k|^2}{\varphi_k}\frac{1 - \mathcal{E}_k(\omega)}{\mathcal{E}_k(\omega)}\right]^{-1},$$

$$\delta \to 0_+ \quad (32)$$

and P stands for the principal value.

In the derivation of Eq. (30), use has been made of the fact that the contributions proportional to $\tilde{D}_k(x)\tilde{D}_k(x+\omega)$ in Eq. (28) will contribute terms to $M(\omega)$ [Eq. (30)], and therefore to the conductivity which are smaller by ω^{-2} with respect to those with $\tilde{D}_k(x)$, i.e., $\chi_k(\alpha_m + \omega_n)$ in Eq. (28) can be replaced by one. The physical reason for this effect is that the

simultaneous excitation of two phonon with a large shift in their energy makes a small contribution to the conductivity in *high frequencies*.

Our final expression for the conductivity is now given by

$$\sigma = \sigma_0(\omega)\left[1 - \frac{2}{3\pi}\frac{e^2}{m\omega^2}F(\omega)\right] \quad (33)$$

where, with Eqs. (14) and (30),

$$F(\omega) = [6\pi^2 m\omega^2/e^2\omega_p{}^2]M(\omega). \quad (34)$$

Equations (33) and (34), with Eq. (32) represent a generalization of the result obtained in I [see Eq. (51) there]. It includes the effect of the finite lifetime of the dressed phonons. In order to obtain a more useful expression for the conductivity we assume that a dressed phonon of momentum $\mathbf{k}$ will have an energy ω_k with a width γ_k much smaller with respect to ω_k (i.e., a long-lived dressed phonon). We therefore obtain, from Eq. (22), for the phonon energy

$$\omega_k{}^2 = \Omega_k{}^2 + \frac{|v_k|^2}{\varphi_k}\,\mathrm{Re}\frac{1 - \mathscr{S}_k(\omega_k)}{\mathscr{S}_k(\omega_k)}, \quad (35)$$

for the phonon lifetime $\Gamma_k(\omega_k)$

$$\Gamma_k{}^{-1}(\omega_k) - \gamma_k(\omega_k) = \frac{1}{2\omega_k}\frac{|v_k|^2}{\varphi_k}\frac{\mathrm{Im}\,\mathscr{S}_k(\omega_k)}{|\mathscr{S}_k(\omega_k)|^2}, \quad (36)$$

and the imaginary part of $\tilde{D}_k(\omega)$ is given by

$$\mathrm{Im}D_k(\omega) = \frac{1}{2\omega_k}\left[\frac{\gamma_k}{(\omega-\omega_k)^2+\gamma_k{}^2} - \frac{\gamma_k}{(\omega+\omega_k)^2+\gamma_k{}^2}\right]. \quad (37)$$

The relation between the conductivity and the relaxation time is given by

$$\sigma(\omega) = i\frac{e^2 n}{m^*}\frac{1}{\omega + i/\tau(\omega)}, \quad (38)$$

where

$$m^* = m[1 + (2e^2/3\pi m\omega^2)\,\mathrm{Re}F(\omega)] \quad (39)$$

and

$$[\tau(\omega)]^{-1} = (2e^2/3\pi m\omega)\,\mathrm{Im}F(\omega). \quad (40)$$

We therefore obtain for the relaxation time

$$[\tau(\omega)]^{-1} = \frac{2}{3mn\omega(2\pi)^3}\int dk\,k^4\frac{|v_k|^2}{\varphi_k}$$

$$\times P\int_{-\infty}^{\infty} dx\,\coth\frac{\beta x}{2}\left[\mathrm{Im}\frac{1}{\mathscr{S}_k(x)}\mathrm{Im}D_k(x+\omega) + \mathrm{Im}\frac{1}{\mathscr{S}_k(x+\omega)}\mathrm{Im}D_k(x)\right], \quad (41)$$

where $\mathrm{Im}D_k(x)$ is given by Eq. (37). This is our general result for the relaxation time which includes the effects of the finite lifetime of the phonons. It is apparent that instead of having a δ function behavior for $\mathrm{Im}D_k(x)$, we obtain here a Lorentzian shape. This amounts to an average of the electron density fluctuation [$\mathrm{Im}(1/\mathscr{S})$] over a range of frequencies of width of the order of $|\gamma_k|$ [see Eq. (36)]. This constitutes a somewhat more realisitic description of the phonons, although it might not change appreciably the numerical values for the relaxation time.

In order to compare our result for the relaxation time with the result given in I, we consider the infinite phonon lifetime case.

(A) If we first assume that the direct response of the phonons to the high-frequency field may be neglected, i.e., we discard the term proportional to $D_k(x+\omega)$ in Eq. (41), we recover the results given in I.

(B) However, if we first carry out the integration over x in Eq. (41) we obtain

$$[\tau(\omega)]^{-1} = \frac{1}{3mn\omega}\frac{1}{4\pi^2}\int dk\,k^4\frac{|v_k|^2}{\varphi_k}\frac{1}{\omega_k}$$

$$\times\left\{\left[\coth\frac{\beta\omega_k}{2}+\coth\frac{\beta}{2}(\omega-\omega_k)\right]\mathrm{Im}\frac{1}{\mathscr{S}_k(\omega-\omega_k)}\right.$$

$$\left.+\left[\coth\frac{\beta\omega_k}{1}-\coth\frac{\beta}{2}(\omega+\omega_k)\right]\mathrm{Im}\frac{1}{\mathscr{S}_k(\omega+\omega_k)}\right\}. \quad (42)$$

For high temperatures (i.e., temperatures above the Debye temperature) Eq. (42) reduces to the same result as in case (A) above.

For low temperatures (i.e., $T \ll T_{\text{Debye}}$) Eq. (42) reads (for positive frequencies and ω above ω_k)

$$[\tau(\omega)]^{-1} = \frac{1}{3mn\omega}\frac{1}{4\pi^2}\int dk\,k^4\frac{|v_k|^2}{\varphi_k}2\,\mathrm{Im}\frac{1}{\mathscr{S}_k(\omega-\omega_k)}. \quad (43)$$

This is to be distinguished from the results of I if ω is in the vicinity of ω_k. However, for $\omega \gg \omega_k$ we again recover the low-temperature result of I, as expected.

It is to be pointed out that in electron-ion systems[8] the results for the high-frequency conductivity can be extended somewhat to frequencies in the vicinity of the collision frequency. However, it is clear, from our present work, that for low temperatures one is not entitled to follow this procedure.

[6] N. Tzoar, Phys. Rev. **132**, 202 (1963).

[7] V. I. Perel and G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **41**, 886 (1961) [English transl.: Soviet Phys.—JETP **14**, 663 (1962)].

[8] J. Dawson and C. Oberman, Phys. Fluids **5**, 517 (1962); A. Ron and N. Tzoar, Phys. Rev. **131**, 1943 (1963).